UNIT-3

3.1 BASIC TRANSFORMATIONS

The basic transformations are :

1. Translation
2. Rotation
3. Scaling

Translation:

A translation moves an object to a different position on the screen.

We translate a two-dimensional point by adding translation distances, t_x, **and** t_y, to the original coordinate position (x, y) to move the point to a new position (x', y') (Fig. 5-1)

$$P = \begin{bmatrix} x_{old} \\ y_{old} \end{bmatrix}$$

$$P' = \begin{bmatrix} x_{new} \\ y_{new} \end{bmatrix}$$

$$T = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$P' = P + T = \begin{bmatrix} x_{old} \\ y_{old} \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

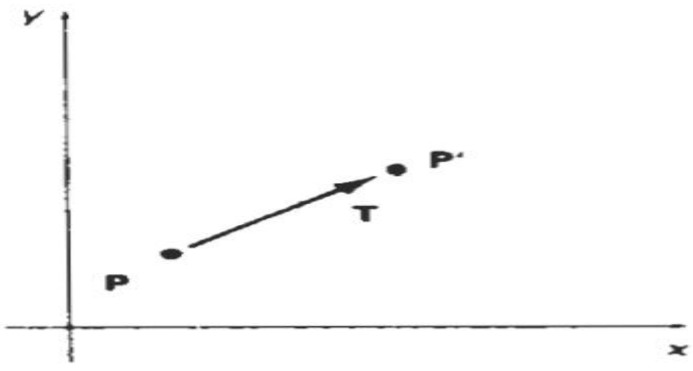


Figure 5-1
Translating a point from position **P** to position **P'** with translation vector **T**.

Problems:

1. **Given a circle C with radius 10 and center coordinates (1, 4). Apply the translation with distance 5 towards X axis and 1 towards Y axis. Obtain the new coordinates of C without changing its radius.**

   Ans: n matrix form, the new center coordinates of C after translation may be obtained as-

$$\begin{bmatrix} X_{new} \\ Y_{new} \end{bmatrix} = \begin{bmatrix} X_{old} \\ Y_{old} \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix}$$

$$\begin{bmatrix} X_{new} \\ Y_{new} \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} X_{new} \\ Y_{new} \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

Thus, New center coordinates of C = (6, 5).

2. Given a square with coordinate points A(0, 3), B(3, 3), C(3, 0), D(0, 0). Apply the translation with distance 1 towards X axis and 1 towards Y axis. Obtain the new coordinates of the square.

Ans: Given-

- Old coordinates of the square = A (0, 3), B(3, 3), C(3, 0), D(0, 0)
- Translation vector = (T_x, T_y) = (1, 1)

<u>For Coordinates A(0, 3)</u>

Let the new coordinates of corner A = (X_{new}, Y_{new}).

Applying the translation equations, we have-

- $X_{new} = X_{old} + T_x = 0 + 1 = 1$
- $Y_{new} = Y_{old} + T_y = 3 + 1 = 4$

Thus, New coordinates of corner A = (1, 4).

<u>For Coordinates B(3, 3)</u>

Let the new coordinates of corner B = (X_{new}, Y_{new}).

Applying the translation equations, we have-

- $X_{new} = X_{old} + T_x = 3 + 1 = 4$
- $Y_{new} = Y_{old} + T_y = 3 + 1 = 4$

Thus, New coordinates of corner B = (4, 4).

<u>For Coordinates C(3, 0)</u>

Let the new coordinates of corner C = (X_{new}, Y_{new}).

Applying the translation equations, we have-

- $X_{new} = X_{old} + T_x = 3 + 1 = 4$
- $Y_{new} = Y_{old} + T_y = 0 + 1 = 1$

Thus, New coordinates of corner C = (4, 1).

<u>For Coordinates D(0, 0)</u>

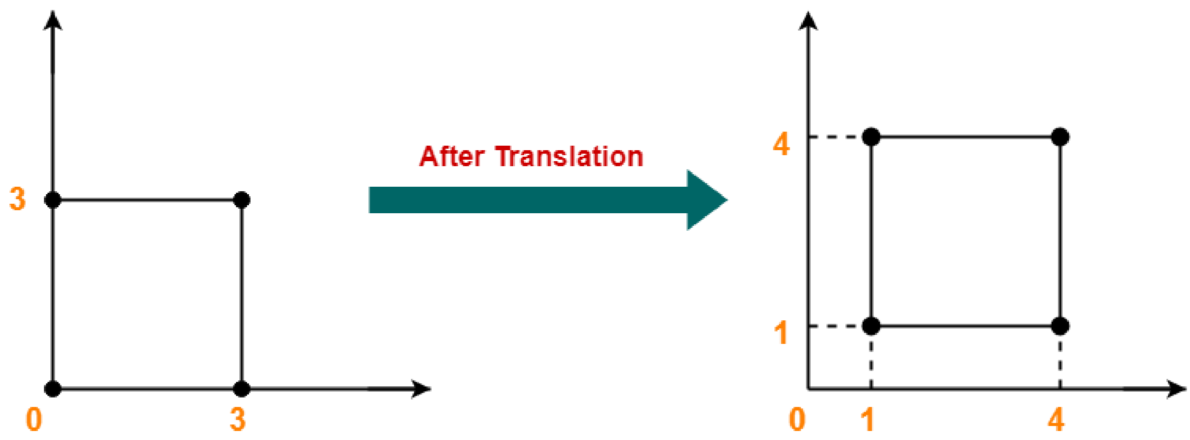
Let the new coordinates of corner D = (X_{new}, Y_{new}).

Applying the translation equations, we have-

- $X_{new} = X_{old} + T_x = 0 + 1 = 1$
- $Y_{new} = Y_{old} + T_y = 0 + 1 = 1$

Thus, New coordinates of corner D = (1, 1).

Thus, New coordinates of the square = A (1, 4), B(4, 4), C(4, 1), D(1, 1).



<u>Exercise Question</u>

Polygon (10, 3), (15,5), (20,3).

Move the polygon with translation amounts of (-5.5, 4)

What are the new coordinates of the polygon after the translation?

<u>Rotation:</u>

A two-dimensional rotation is applied to an object by repositioning it along a circular path in the xy- plane.

To generate a rotation, we specify a rotation angle θ and the position (x_r, y_r) of the rotation point (or pivot point) about which the object is to be rotated (Fig. 5-3)

Positive values for the rotation angle define counter clockwise rotations about the pivot point, as in Fig. 5-3, and negative values rotate objects in the clockwise direction.

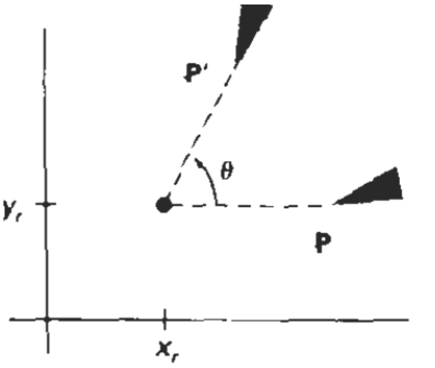


Figure 5-3
Rotation of an object through angle θ about the pivot point (x_r, y_r).

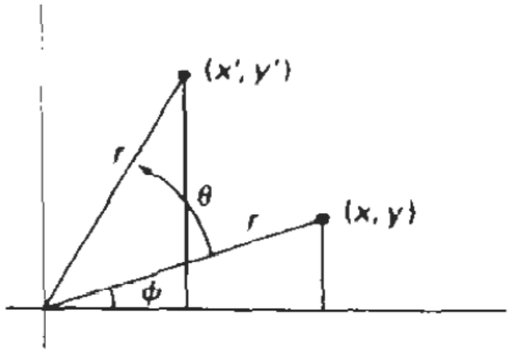


Figure 5-4
Rotation of a point from position (x, y) to position (x', y') through an angle θ

$$x' = r\cos(\phi + \theta) = r\cos\phi\cos\theta - r\sin\phi\sin\theta$$
$$y' = r\sin(\phi + \theta) = r\cos\phi\sin\theta + r\sin\phi\cos\theta \qquad (5\text{-}4)$$

The original coordinates of the point in polar coordinates are

$$x = r\cos\phi, \qquad y = r\sin\phi \qquad (5\text{-}5)$$

Substituting expressions 5-5 into 5-4,

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta \qquad (5\text{-}6)$$

$$\mathbf{P}' = \mathbf{R} \cdot \mathbf{P}$$

where the rotation matrix is

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{P}'^T = (\mathbf{R} \cdot \mathbf{P})^T = \mathbf{P}^T \cdot \mathbf{R}^T$$

Rotation of a point about an arbitrary pivot position is illustrated in Fig. 5-5. Using trigonometric relationships in this figure, we can generalize Eqs. 5-6 to obtain the transformation equations for rotation of a point about any specified rotation position (x_r,y_r)

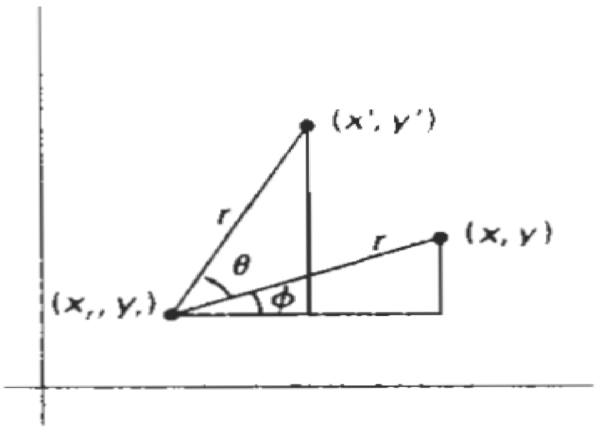


Figure 5-5
Rotating a point from position (x, y) to position (x', y') through an angle θ about rotation point (x_r, y_r).

$$x' = x_r + (x - x_r)\cos\theta - (y - y_r)\sin\theta$$
$$y' = y_r + (x - x_r)\sin\theta + (y - y_r)\cos\theta \qquad (5\text{-}9)$$

Scaling:

- A scaling transformation alters the size of an object.
- This operation can be carried out for polygons by multiplying the coordinate values (x, y) of each vertex by scaling factors s, and s, to produce the transformed coordinates (x', y'):

$$x' = x.\ s_x$$

$$y' = y.\ s_y$$

"Scaling factor s_x, scales objects in the x direction, while s_y scales in the y direction."

The transformation equations 5-10 can also be written in the matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \qquad (5\text{-}11)$$

or

$$\mathbf{P}' = \mathbf{S} \cdot \mathbf{P} \qquad (5\text{-}12)$$

where **S** is the 2 by 2 scaling matrix in Eq. 5-11.

Example of Scaling:

Figure 5-6
Turning a square (a) into a rectangle (b) with scaling factors $s_x = 2$ and $s_y = 1$.

- We can control the location of a scaled object by choosing a position, called the **fixed point**, that is to remain unchanged after the scaling transformation
- Coordinates for the fixed point (x_f, y_f) can be chosen as one of the vertices, the object centroid, or any other position (Fig. 5-8). A polygon is then scaled relative to the fixed point by scaling the distance from each vertex to the fixed point. For a vertex with coordinates (x,y), the scaled coordinates (x', y ') are calculated as

$$x' = x_f + (x - x_f)s_x, \qquad y' = y_f + (y - y_f)s_y \tag{5-13}$$

We can rewrite these scaling transformations to separate the multiplicative and additive terms:

$$\begin{aligned} x' &= x \cdot s_x + x_f(1 - s_x) \\ y' &= y \cdot s_y + y_f(1 - s_y) \end{aligned} \tag{5-14}$$

where the additive terms $x_f(1 - s_x)$ and $y_f(1 - s_y)$ are constant for all points in the object.

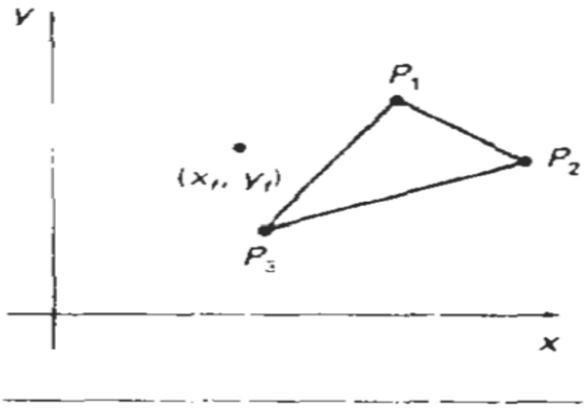


Figure 5-8
Scaling relative to a chosen fixed point (x_f, y_f). Distances from each polygon vertex to the fixed point are scaled by transformation equations 5-13.

2. MATRIX REPRESENTATIONS AND HOMOGENEOUS COORDINATES

A. Translation Matrix in Homogeneous Coordinates:

Coordinates are represented by three element column vectors, and transformation operations are written as 3 by 3 matrices.

For translation, we have $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

$$\mathbf{P}' = \mathbf{T}(t_x, t_y) \cdot \mathbf{P}$$

- ⇨ $x' = x + t_x$
- ⇨ $y' = y + t_y$

⇨ z' =1

B. Rotation Matrix in Homogeneous Coordinates:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} cos\theta - sing\theta & 0 \\ sin\theta \quad cos\theta & 1 \\ 0 \quad\quad 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{P'} = \mathbf{R}(\theta) \cdot \mathbf{P}$$

⇨ x' = xcosθ - ysinθ
⇨ y' = xsinθ + ycosθ
⇨ z' =1

C. Scaling Matrix in Homogeneous Coordinates:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{P'} = \mathbf{S}(s_x , s_y) \cdot \mathbf{P}$$

⇨ x' = xS_x
⇨ y' = yS_y
⇨ z' =1

3. COMPOSITE TRANSFORMATIONS

We can set up a matrix for any sequence of transformations as a **composite transformation** matrix by calculating the matrix product of the individual transformations.

Forming products of transformation matrices is often referred to as a concatenation, or composition, of matrices.

For column-matrix representation of coordinate positions, we form composite transformations by multiplying matrices in order from right to left.

A. TRANSLATION COMPOSITE MATRICES

If two successive translation vectors (t_{x1},t_{y1}) and (t_{x2},t_{y2}) are applied to a coordinate position P, the final transformed location P' is calculated as

$$\boldsymbol{P'} = \boldsymbol{T}(\boldsymbol{t_{x2},t_{y2}}).\{\boldsymbol{T}(\boldsymbol{t_{x1},t_{y1}}).\boldsymbol{P}\}$$

$$\begin{bmatrix} 1 & 0 & t_{x2} \\ 0 & 1 & t_{y2} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & t_{x1} \\ 0 & 1 & t_{y1} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_{x1} + t_{x2} \\ 0 & 1 & t_{y1} + t_{y2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}(t_{x2}, t_{y2}) \cdot \mathbf{T}(t_{x1}, t_{y1}) = \mathbf{T}(t_{x1} + t_{x2}, t_{y1} + t_{y2})$$

Which demonstrates that two successive translations are additive

B. ROTATION COMPOSITE MATRICES

Two successive rotations applied to point P produce the transformed position

$$\begin{aligned} \mathbf{P}' &= \mathbf{R}(\theta_2) \cdot \{\mathbf{R}(\theta_1) \cdot \mathbf{P}\} \\ &= \{\mathbf{R}(\theta_2) \cdot \mathbf{R}(\theta_1)\} \cdot \mathbf{P} \end{aligned} \tag{5-26}$$

By multiplying the two rotation matrices, we can verify that two successive rotations are additive:

$$\mathbf{R}(\theta_2) \cdot \mathbf{R}(\theta_1) = \mathbf{R}(\theta_1 + \theta_2) \tag{5-27}$$

so that the final rotated coordinates can be calculated with the composite rotation matrix as

$$\mathbf{P}' = \mathbf{R}(\theta_1 + \theta_2) \cdot \mathbf{P} \tag{5-28}$$

C. SCALING COMPOSITE MATRICES

Scalings

Concatenating transformation matrices for two successive scaling operations produces the following composite scaling matrix:

$$\begin{bmatrix} s_{x2} & 0 & 0 \\ 0 & s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_{x1} & 0 & 0 \\ 0 & s_{y1} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_{x1} \cdot s_{x2} & 0 & 0 \\ 0 & s_{y1} \cdot s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5-29}$$

or

$$\mathbf{S}(s_{x2}, s_{y2}) \cdot \mathbf{S}(s_{x1}, s_{y1}) = \mathbf{S}(s_{x1} \cdot s_{x2}, s_{y1} \cdot s_{y2}) \tag{5-30}$$

D. GENERAL PIVOT-POINT ROTATION

With a graphics package that only provides a rotate function for revolving objects about the coordinate origin, we can generate rotations about any selected pivot point (x, y,) by performing the following sequence of **translate-rotate translate operations**:

1. Translate the object so that the pivot-point position is moved to the coordinate origin.
2. Rotate the object about the coordinate origin.
3. Translate the object so that the pivot point is returned to its original position

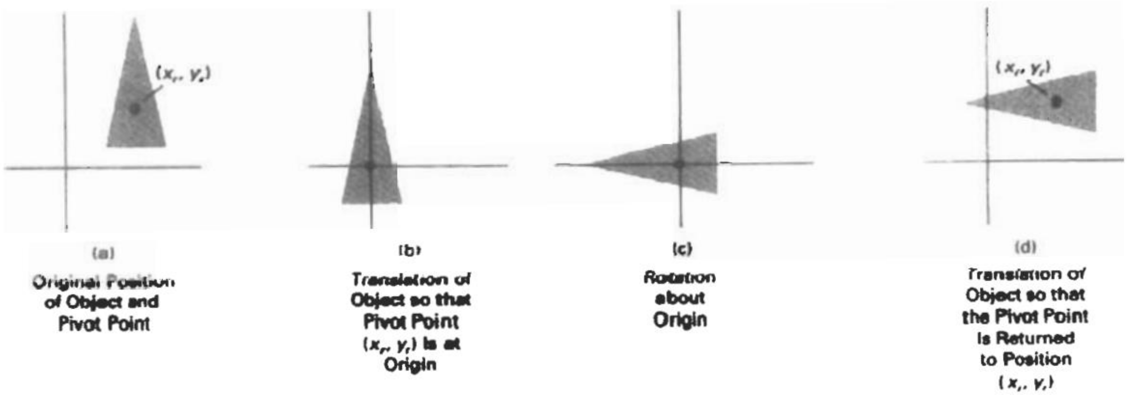


Figure 5-9
A transformation sequence for rotating an object about a specified pivot point using the rotation matrix $\mathbf{R}(\theta)$ of transformation 5-19.

$$\begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & x_r(1-\cos\theta) + y_r\sin\theta \\ \sin\theta & \cos\theta & y_r(1-\cos\theta) - x_r\sin\theta \\ 0 & 0 & 1 \end{bmatrix}$$

which can be expressed in the form

$$\mathbf{T}(x_r, y_r) \cdot \mathbf{R}(\theta) \cdot \mathbf{T}(-x_r, -y_r) = \mathbf{R}(x_r, y_r, \theta)$$

E. GENERAL FIXED-POINT SCALING

1. Translate object so that the fixed point coincides with the coordinate origin.
2. Scale the object with respect to the coordinate origin.
3. Use the inverse translation of step 1 to return the object to its original position.

Concatenating the matrices for these three operations produces the required scaling matrix

$$\begin{bmatrix} 1 & 0 & x_f \\ 0 & 1 & y_f \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_f \\ 0 & 1 & -y_f \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & x_f(1 - s_x) \\ 0 & s_y & y_f(1 - s_y) \\ 0 & 0 & 1 \end{bmatrix}$$

or

$$\mathbf{T}(x_f, y_f) \cdot \mathbf{S}(s_x, s_y) \cdot \mathbf{T}(-x_f, -y_f) = \mathbf{S}(x_f, y_f, s_x, s_y)$$

This transformation is automatically generated on systems that provide a scale function that accepts coordinates for the fixed point.

4. OTHER TRANSFORMATIONS

A. Reflection Transformations

A reflection is a transformation that produces a mirror image of an object. The mirror image for a two-dimensional reflection is generated relative to an axis of reflection by rotating the object 180" about the reflection axis.

Reflection about the line $y = 0$, the x axis, is accomplished with the transformation matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

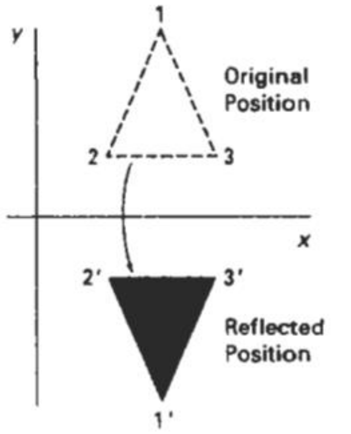


Figure 5-16
Reflection of an object about the x axis.

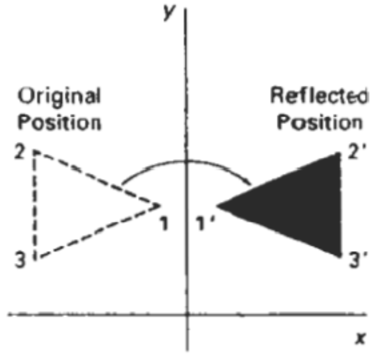


Figure 5-17
Reflection of an object about the y axis.

A reflection about the y axis flips x coordinates while keeping y coordinates the same. The matrix for this transformation is

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

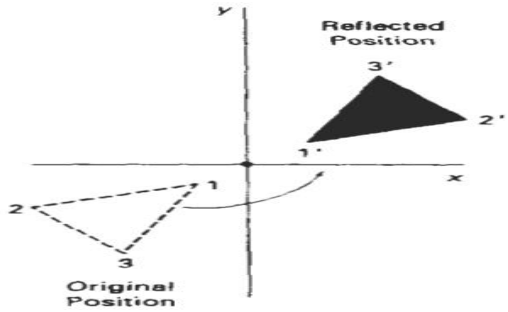


Figure 5-18
Reflection of an object relative to an axis perpendicular to the xy plane and passing through the coordinate origin.

We flip both the x and y coordinates of a point by reflecting relative to an axis that is perpendicular to the xy plane and that passes through the coordinate origin. This transformation, referred to as a reflection relative to the coordinate origin, has the matrix representation:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

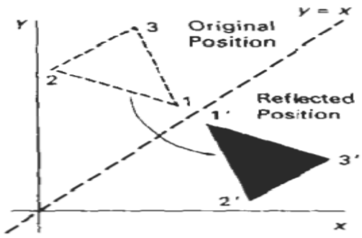


Figure 5-20
Reflection of an object with respect to the line $y = x$.

To obtain a transformation matrix for reflection about the diagonal $y = -x$, we could concatenate matrices for the transformation sequence: (1) clockwise rotation by 45°, (2) reflection about the y axis, and (3) counterclockwise rotation by 45°. The resulting transformation matrix is

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

B. Shear Transformation

A transformation that distorts the shape of an object such that the transformed caused shape appears as to slide over if each the object other were is called composed a shear

An x-direction shear relative to the x axis is produced with the transformation matrix

$$\begin{bmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which transforms coordinate positions as

$$x' = x + sh_x \cdot y, \qquad y' = y$$

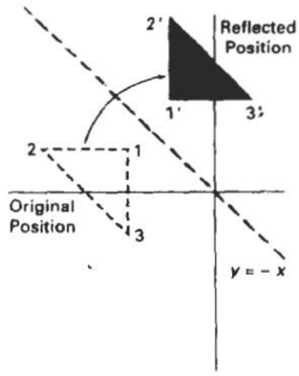


Figure 5-22
Reflection with respect to the line $y = -x$.

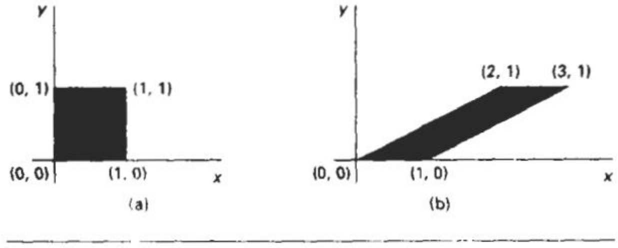


Figure 5-23
A unit square (a) is converted to a parallelogram (b) using the x-direction shear matrix 5-53 with $sh_x = 2$.

Any real number can be assigned to the shear parameter sh_x. A coordinate position (x, y) is then shifted horizontally by an amount proportional to its distance (y value) from the x axis ($y = 0$). Setting sh_x to 2, for example, changes the square in Fig. 5-23 into a parallelogram. Negative values for sh_x shift coordinate positions to the left.

We can generate x-direction shears relative to other reference lines with

$$\begin{bmatrix} 1 & sh_x & -sh_x \cdot y_{ref} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with coordinate positions transformed as

$$x' = x + sh_x(y - y_{ref}), \qquad y' = y$$

A y-direction shear relative to the line $x = x_{ref}$ is generated with the transformation matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ sh_y & 1 & -sh_y \cdot x_{ref} \\ 0 & 0 & 1 \end{bmatrix}$$

which generates transformed coordinate positions

$$x' = x, \qquad y' = sh_y(x - x_{ref}) + y$$

This transformation shifts a coordinate position vertically by an amount proportional to its distance from the reference line $x = x_{ref}$. Figure 5-25 illustrates the conversion of a square into a parallelogram with $sh_y = 1/2$ and $x_{ref} = -1$.

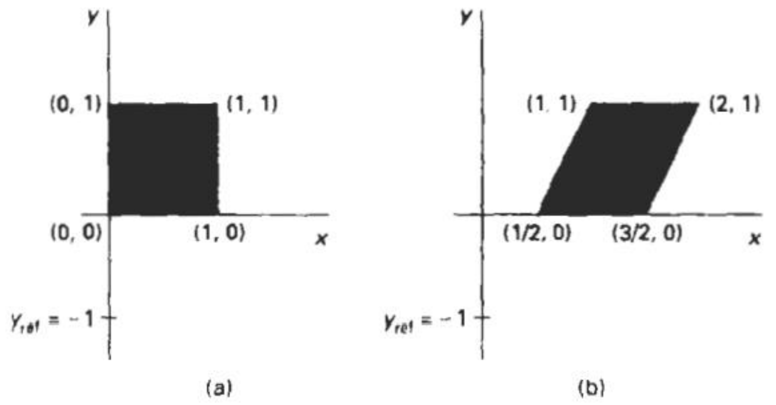


Figure 5-24
A unit square (a) is transformed to a shifted parallelogram (b) with $sh_x = 1/2$ and $y_{ref} = -1$ in the shear matrix 5-55.